VI 4-Manifolds and Surgery

A. Handlebody Theory and Morse Functions

an n-dimensional <u>k-handle</u> is

$$h^k = D^k \times D^{n-k}$$

set
$\partial_- h^k = (\partial D^k) \times D^{n-k}$ — <u>attaching region</u>

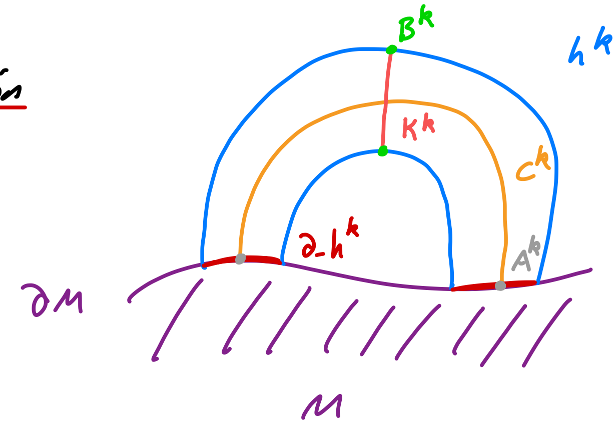
$\partial_+ h^k = D^k \times \partial D^{n-k}$

$A^k = \partial D^k \times \{0\}$ — <u>attaching sphere</u>

$C^k = D^k \times \{0\}$ — <u>core</u>

$K^k = \{0\} \times D^{n-k}$ — <u>co-core</u>

$B^k = \{0\} \times \partial D^{n-k}$ — <u>belt sphere</u>



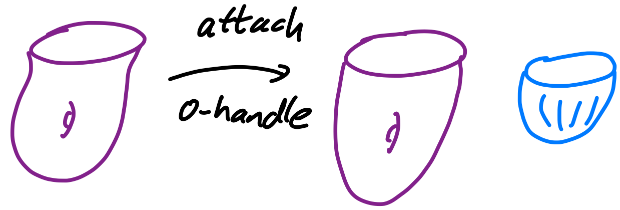
given an n-manifold M and an embedding

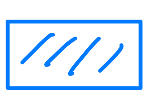
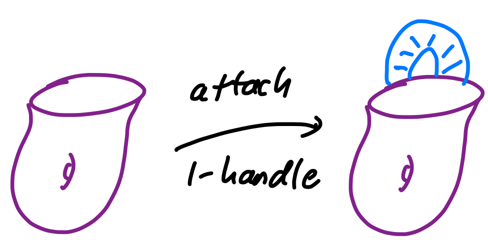
$$\phi : \partial_- h^k \to \partial M$$

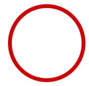
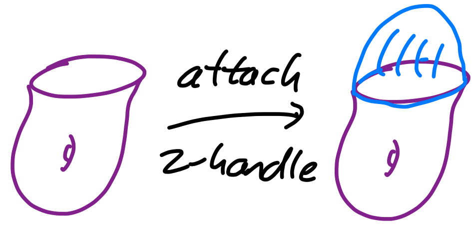
we attach h^k to M by forming the identification space

$$M \amalg h^k \Big/ (x \in \partial_- h^k) \sim (\phi(x) \in \partial M)$$

e.g. dimension 2:

<u>k = 0</u>:  $\partial_- h^0 = \emptyset$ so 

<u>k = 1</u>:  $\partial_- h^1 =$  so 

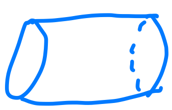
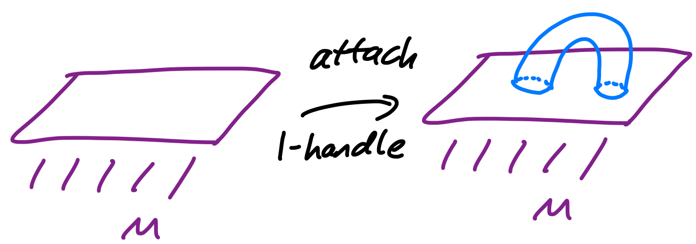
$\underline{k=2}$:  $\partial_- h^2 =$  so 

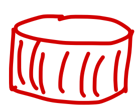
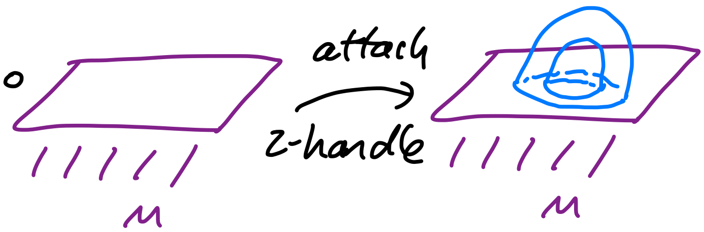
Remark:

1) In all dimensions attaching a 0-handle is just taking a disjoint union with D^n
2) In all dimensions n, attaching an n-handle is just "capping off" an S^{n-1} boundary component

dimension 3:

$\underline{k=0}$: ✓

$\underline{k=1}$:  $\partial_- h^1 =$  so 

$\underline{k=2}$:  $\partial_- h^2 =$  so 

$\underline{k=3}$: ✓

Remark: Note when attaching a handle one has a manifold with "corners", there is a standard way to smooth them out (see Wall "Differential Topology")

exercise:

1) if $\phi_0, \phi_1 : \partial_- h^k \to \partial M$ are isotopic, then the result of attaching a handle to M via ϕ_0 is diffeomorphic to attaching a handle to M via ϕ_1

2) the isotopy class of $\phi: \partial_- h^k \to M$ is determined by

1) isotopy class of $\phi|_{A^k}$ $(A^k = S^{k-1} \times \{0\})$

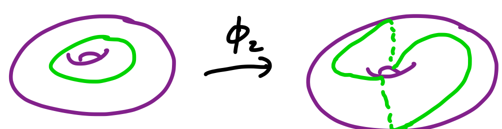
(i.e. a S^{k-1} knot in ∂M)

2) the "framing" of the normal bundle of $\phi(A^k)$ given by $\phi|_{\partial_- h^k = A^k \times D^{n-k}}$

i.e. an identification of $\nu(\phi(A^k))$ with $S^{k-1} \times D^{n-k}$

e.g. notice that $S^1 \times D^2$ has an integers worth of framings

$$\begin{aligned} S^1 \times D^2 &\xrightarrow{\phi_n} S^1 \times D^2 \\ (\phi, (r, \theta)) &\longmapsto (\phi, (r, \theta + n\phi)) \end{aligned}$$



3) more generally show the framings on a k-dimensional sphere in Y^n is in one-to-one correspondence with $\pi_k(O(n-k))$

dim of normal bundle fiber

So we see to attach an n-dimensional k-handle, one must specify

1) an S^{k-1} knot in ∂M and

2) "elt" of $\pi_{k-1}(O(n-k))$

really to get such an element need a canonical "zero" framing

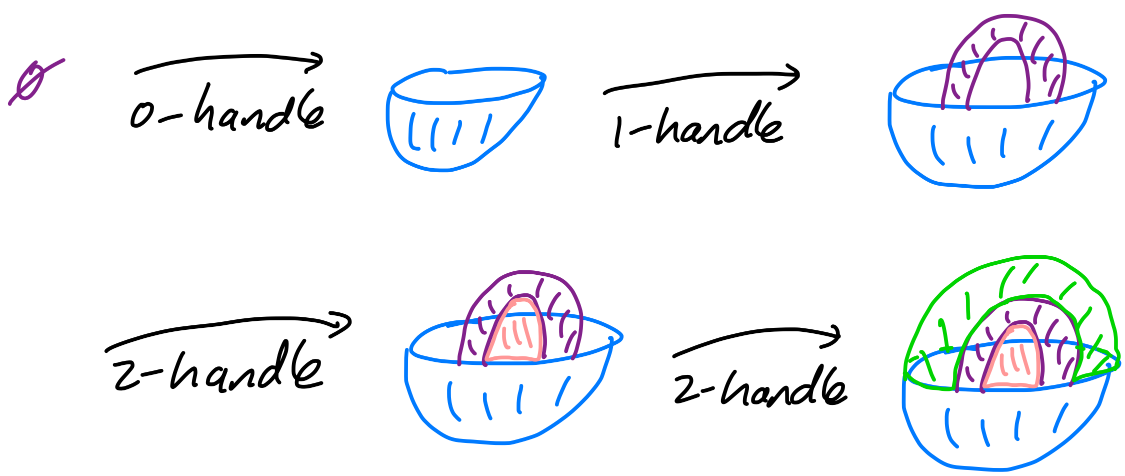
A handle decomposition of an n-manifold M is a sequence of manifolds $M_0, M_1, \dots, M_\ell$ such that

1) $M_0 = \emptyset$ and $M_\ell \cong M$
2) M_{i+1} is obtained from M_i by a k-handle attachment for some k

example: handle decompositions of S^2

1)

2)



Thm:

$$\boxed{\text{Any smooth compact manifold has a handle decomposition}}$$

a brief sketch of the proof goes as follows

recall a Morse function

$$f: M \to \mathbb{R}$$

is a function all of whose critical points are non-degenerate, that is if $p \in M$ is a critical point, then in local coordinates about p, the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is inverteble

<u>exercise:</u> 1) Show p is a non-degenerate critical point of $f \iff df$ is transverse to the zero section of T^*M at $df(p)$

2) Every function $f: M \to \mathbb{R}$ can be perturbed to be a Morse function

3) If p is a non-degenerate critical point of $f: M \to \mathbb{R}$ then $\exists$ coordinates about p such that f takes the form

Fundamental lemma of Morse theory

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

k is called the <u>index of p</u>

<u>Main Th$^{\underline{m}}$ of Morse theory:</u>

let $f: M \to \mathbb{R}$ be a Morse function

I) if $[a,b]$ contains no critical values then

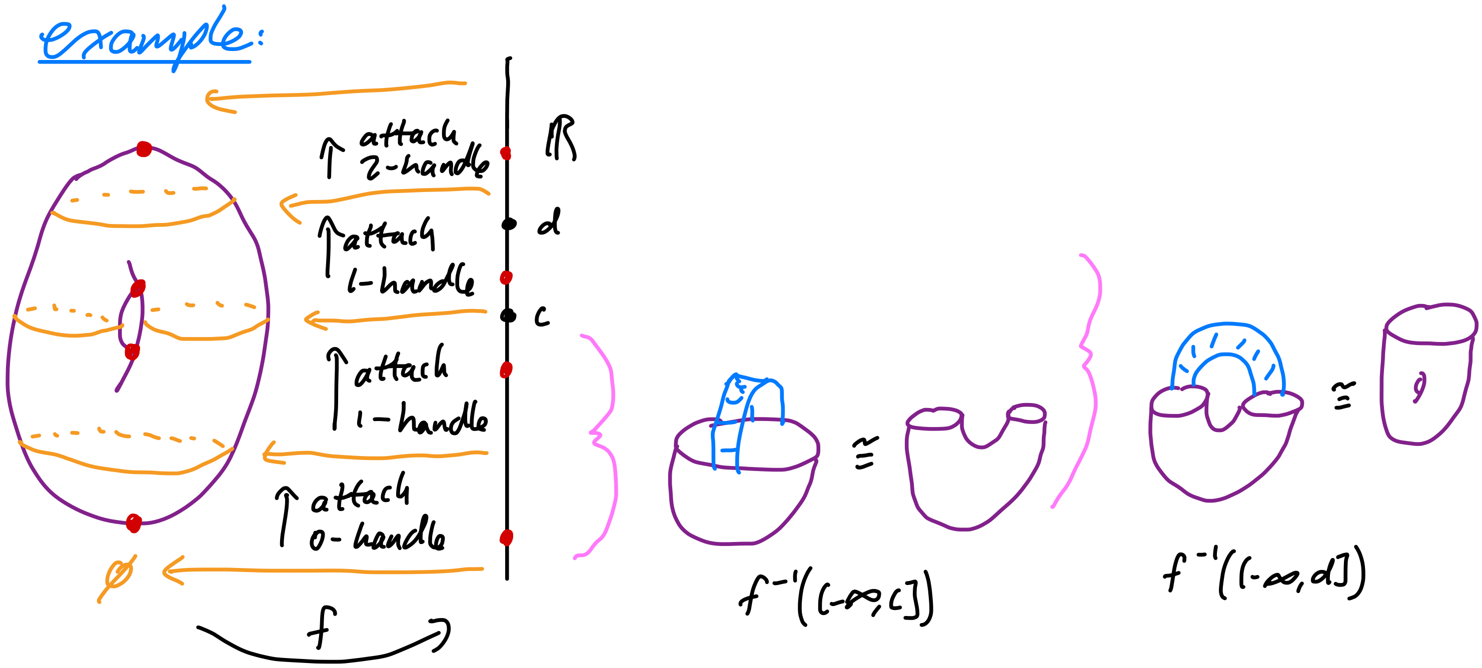
$$f^{-1}([a,b]) \cong f^{-1}(a) \times [a,b]$$

manifold since a reg. value

II) if $\exists!$ critical point $p \in f^{-1}([a,b])$ s.t. $f(p) \in (a,b)$ then $f^{-1}([a,b])$ is obtained from $f^{-1}(a) \times [a, a+\varepsilon]$ by attaching a k-handle to $f^{-1}(a) \times \{a+\varepsilon\}$

index of p

example:



Remark: handle decomposition theorem clearly follows

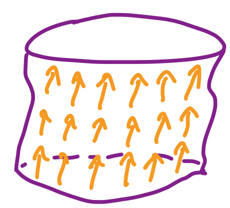
Idea of Proof of Main Th^m:

I) let $\Phi_t : M \to M$ be the (normalized) gradient flow of f

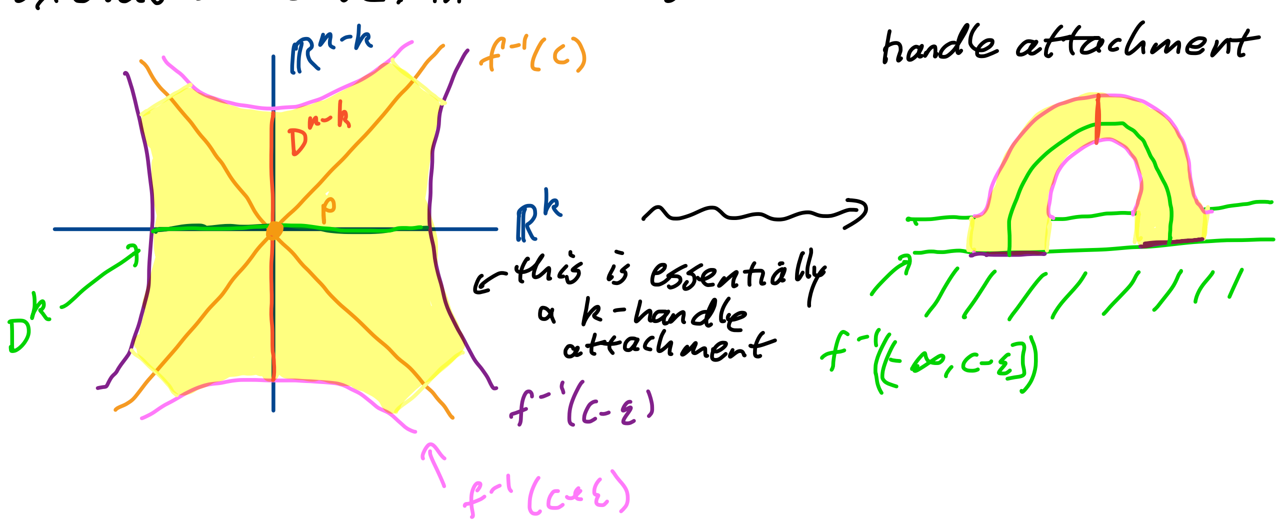
then $f^{-1}(a) \times [a,b] \to M$

$(p,t) \longmapsto \Phi_{t-a}(p)$

is an embedding onto $f^{-1}([a,b])$



II) let U be nbhd about p where f has the form as in exercis 3 above. In U we see



<u>exercise</u>: finish proof of II)

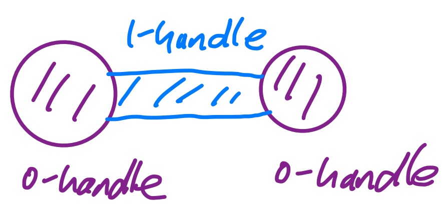
<u>Facts (exercises)</u>:

1) In a handle decomposition, can assume handles are attached in order of non-decreasing index

   <u>Hint</u>: if h^k attached after h^ℓ and $k \le \ell$, show you can make the attaching sphere of h^k disjoint from the belt sphere of h^ℓ

2) if the attaching sphere of h^{k+1} intersects the belt sphere of h^k exactly once and transversely, then $M \cup h^k \cup h^{k+1} \cong M$

3) if M is connected then it has a handle decomp. with only one 0-handle

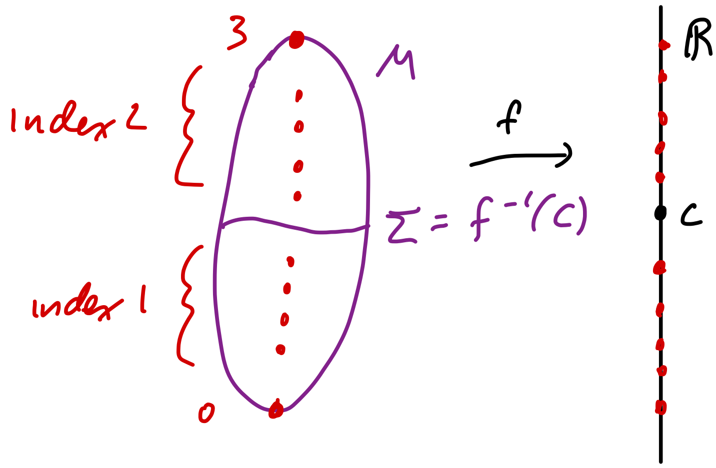


4) if $f: M^n \to \mathbb{R}$ is Morse then so is $-f$

   and index k for $f \longleftrightarrow$ index $n-k$ for $-f$

   (so if $\partial M = \emptyset$ then can assume only one n-handle and if $\partial M \neq \emptyset$ then can assume <u>no</u> n-handles)

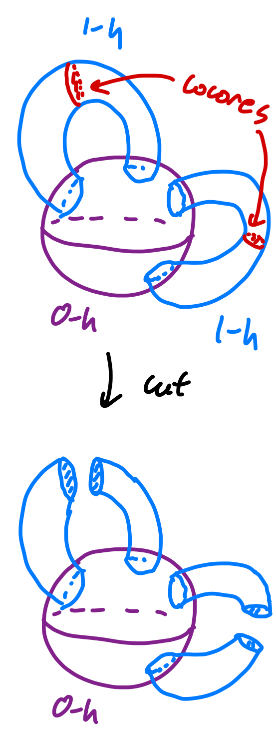
example: M a closed, connected, oriented 3-manifold
$f: M^3 \to \mathbb{R}$ a Morse function with
one index 0 critical point
one index 3 " "



$f^{-1}(c)$ = surface of genus g

note: $M_c = f^{-1}((-\infty, c]) =$ 0-handle $\cup$ 1-handles
is a "handlebody" in the sense of section I:

$M_c \setminus$ co-cores of 1-handles
$\shortparallel$
0-handle ($\cong$ 3 ball)
so follows from lemma I.1



similarly $\overline{M - M_c} = f^{-1}([c, \infty))$
$= (-f)^{-1}((-\infty, -c])$
$=$ handle body too

so Σ is a Heegaard splitting!
New proof of Th$^{\underline{m}}$ I.3 using Morse Theory.

B 4-manifolds

X a connected 4-manifold

so X has a handlebody structure with one 0-handle

so other handles attached to $S^3 = \partial(D^4) = \mathbb{R}^3 \cup \{\infty\}$

that is, we can draw them in $\mathbb{R}^3$

<u>1-handle</u>: $h^1 = D^1 \times D^2$ attaching region $\partial D^1 \times D^2 = S^0 \times D^2$

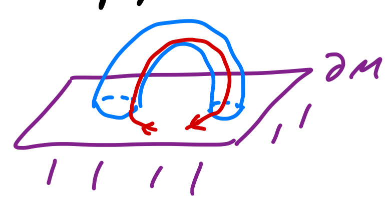
framing $\in \pi_0(O(3)) \cong \mathbb{Z}/2$

<u>exercise</u>: one of these framings gives a non-orientable manifold

so if X orientable attaching h^1 determined by image of $\partial_- h^1$

(if walking around boundary and walk in one D^3 will pop out other one

in 3D



)

now $h^0 \cup h^1 = D^4 \cup_\sim (D^1 \times D^3) = (D^1 \times D^3) \cup_\sim (D^1 \times D^3)$

glue $\partial_- h^1$ to $\partial D^1 \times D^3$

$= S^1 \times D^3$

and $h^0 \cup m h^1 = \natural_m S^1 \times D^3$

where $\natural$ is <u>boundary sum</u>

that is, given X_1 and X_2

let D_1, D_2 be D^3's in ∂X_1 and ∂X_2

$$X_1 \natural X_2 = X_1 \cup_{D_1 = D_2} X_2$$

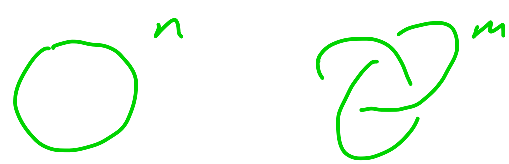
<u>2-handles:</u> $h^2 = D^2 \times D^2$ attaching region $\partial_- h^2 = S^1 \times D^2$

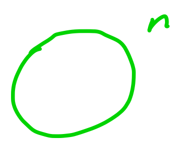
framing in $\pi_1(SO(2)) \cong \mathbb{Z}$

so we need to specify a knot K in $\partial((0\text{-}h) \cup (1\text{-}h)s)$

and a framing on K

(identified with $\mathbb{Z}$ using a Seifert surface for K if K is null-homologous)

example: 

what is  ?

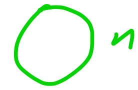
$$0\text{-}h \cup 2\text{-}h$$
$$\parallel$$
$$(D^2 \times D^2) \cup (D^2 \cup D^2)$$

glue $A = S^1 \times D^2$ in 2^{nd} factor to

$\underbrace{S^1 \times D^2 \text{ in } 1^{st} \text{ factor}}_{\text{nbhd of unknot in } \partial D^4}$

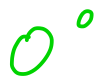
if one glues $S^1 \times D^2 \to S^1 \times D^2$ by identity

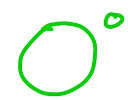
then get $S^2 \times D^2$

exercise: in general get D^2-bundle over S^2 an the Euler class of  is n

(recall D^2-bundles over S^2 are in one-to-one correspondence with $\mathbb{Z}$ via Euler class)

note: $\partial(0\text{-h} \cup 1\text{-h}) = S^1 \times S^2$ and

$\partial($  $) = S^1 \times S^2$

so if we are only interested in ∂X then can replace  with 

exercise: more generally if you see

this has same boundary as

$\therefore$ if $M^3 = \partial X^4$ and $X^4 = 0\text{-h} \cup (1\text{-h})\text{'s} \cup (2\text{-h})\text{'s}$ then $\exists\, X'$ a 4-manifold with only 0- and 2-handles such that $\partial X' = M$

Fact: if a 4-mfd is closed then there is a unique way to attach 3- and 4-handles

to get the closed manifolds
so we don't need to keep track of them!

note: if X is a 4-manifold such that $\partial X = M^3$
then from above we can assume X has
no 4-handles and we can replace 1-handles
with 2-handles, moreover by "turning upside
down" (replace f by $-f$) can replace 3-handles
by 2-handles too.

That is if $M = \partial(\text{4-manifold})$ then M is the
boundary of a 4-manifold with only a 0-handle
and 2-handles

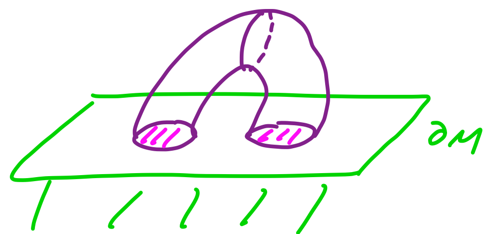
what happens to the boundary of an n-manifold when you
attach a k-handle?

lemma 1:

if $X' = X \cup k\text{-handle}$ then $\partial X' = \partial X$ with a nbhd of
the attaching sphere removed and $\partial_+ h^k = D^k \times S^{n-k-1}$
glued in via attaching map restricted to $\partial(\partial_- h^k) = \partial(\partial_+ h^k)$
(ie remove $S^{k-1} \times D^{n-k}$ reglue $D^k \times S^{n-k-1}$ this is called
surgery on the S^{k-1} in ∂X)

if $\dim X = 4$ and h^2 is attached along an $S^1 \subset \partial X$ with
framing n, then $\partial X' = \partial X$ after n-framed Dehn surgery on K

example: in 3D attach a 1-handle



note the image of $\partial_- h^1$ becomes part of the interior of X'

and we glue a copy of $\partial_+ h^1$ to $\partial X - (\text{nbhd } \partial_- h^1)$ along

$\partial(\partial_+ h^1) = \partial D^1 \times \partial D^2 = S^0 \times S^1$

Proof: assume h^k is glued via $\phi : \partial_- h^k \to \partial X$

then as above $\phi(\partial_- h^k) = (\text{nbhd attaching sphere})$ is

removed from ∂X (it becomes part of the interior)

and we add $\partial_+ h^k$ to get $\partial X'$

we glue $\partial_+ h^k = D^k \times \partial D^{n-k} = D^k \times S^{n-k-1}$ to $\partial X - \phi(\partial_- h^k)$

via $\phi|_{\partial(\partial_+ h^k)} : S^{k-1} \times S^{n-k-1} \to \overline{\partial X - \phi(\partial_- h^k)}$

exercise: convince yourself of last part of lemma

(hopefully clear) ▦

note: from Section I and V we know any oriented 3-manifold is obtained from S^3 by Dehn surgery on a link with integer coefficients, thus it is the boundary of a 4-manifold!

We can prove this independent of Section I and therefore give an alternate proof that 3-mfds can be obtained from S^3 via Dehn surgery!

(Wallace)

Th^{m}2:

> any closed orientable 3-manifold is the boundary of a 4-manifold built with only a 0-handle an 2-handles ($\Rightarrow$ obtained from S^3 via Dehn surgery)

Proof: the Whitney immersion theorem say an orientable n-manifold immerses in S^{2n-1}

let $i: M^3 \to S^5$ be an immersion in S^5 we can isotop i so it is self-transverse i.e. the double points of $i(M)$ are $\amalg \gamma_i$, each γ_i an S^1 in $i(M)$

Claim: 1) $\exists$ a 4-manifold X such that $\partial X = M \cup -M'$ where M' is embedded in $\mathbb{R}^5$

2) if M' is embedded in $\mathbb{R}^5$ then there is an embedded "Seifert submanifold" W in $\mathbb{R}^5$ such that $\partial W = M'$

given claims $M = \partial(\underbrace{X \cup W'}_{X'})$ and from above X' can be assumed to have a 0-handle and 2-handles

$\therefore$ lemma 1 $\Rightarrow$ $\partial X' = M$ is obtained by surgery on a link in S^3.

for Claim 1: consider the double points of $i: M \to S^5$

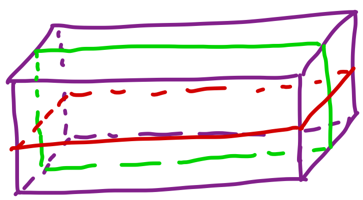
if $i^{-1}(\Gamma_1) = \gamma_1 \cup \gamma_1'$ is disconnected

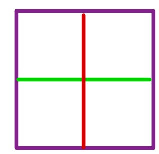
then $N(\gamma_1) = \gamma_1 \times D^2$ and

$N(\gamma_1') = \gamma_1' \times D^2$

so $N(\Gamma_1) = \Gamma_1 \times D^4 \cong \underbrace{\gamma_1 \times D^2}_{N(\gamma_1)} \times D^2 = \gamma_1 \times D^4$

$N(\gamma_1')$



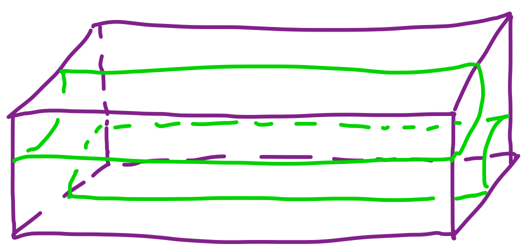


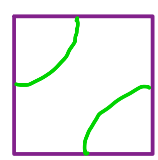
for each $p \in \gamma_1$ replace 2 disks in D^4 with

an annulus $A = S^1 \times [1,2]$

with $A \times \{1\} = \gamma_1 \times \partial D^2$ in $N(\gamma_1)$

and $A \times \{2\} = \gamma_1 \times \partial D^2$ in $N(\gamma_1')$





so let $M' = \overline{M - (N(\gamma \cup \gamma'))} \cup (A \times S^1)$

<u>note</u>: $M' \looparrowright S^5$ has one less double point set

and $X = M \times [0,1] \cup (D^2 \times [1,2] \times S^1)$ where

$D^2 \times \{1\} \times S^1$ glued to $N(\gamma')$ and

$D^2 \times \{2\} \times S^1$ " " $N(\gamma^4)$ in $M \times \{1\}$

then $\partial X = -M \cup M'$

exercise: if $i^{-1}(\Gamma_1)$ is connected give a similar construction

for Claim 2: this follows from

Th^{m}3:

if M is a connected, closed, oriented m-manifold smoothly embedded in an oriented W^{m+2} and

$$[M] = 0 \text{ in } H_m(W)$$

then $\exists$ an embedded $\Sigma^{m+1} \subset W^{m+2}$ s.t.

$$\partial \Sigma = M$$

ultimate generalization of a "Seifert surface"!

for this we need

lemma 4:

if M is a connected, closed, oriented m-manifold smoothly embedded in an oriented W^{m+2} and

$$[M] = 0 \text{ in } H_m(W)$$

then it has a trivial normal bundle

in particular M has a nbhd $\cong M \times D^2$

Proof: recall the Euler class of an oriented vector bundle $\begin{matrix} E \\ \downarrow \\ M^m \end{matrix}$ with fiber $\mathbb{R}^k$ is the Poincaré dual $e(E)$ of $[\sigma^{-1}(\text{zero section})] \in H_{m-k}(M)$

where $\sigma: M \to E$ is a section that is transverse to the zero section

exercise:

1) show this is well-defined i.e. independent of σ.

2) If $k=2$, then $e(E)=0 \iff E$ has a non-zero section

Warning: Not true in general! only get $(\Leftarrow)$

Hint: $e(E)$ is the "obstruction" to E having a non-zero section over the k-skeleton but for $k=2$ no other obstructions

now since the normal bundle $\mathbb{R}^2 \to \nu(M) \xrightarrow{\pi} M$ is an orientable bundle (since M and W are oriented) it will be trivial if $\exists$ a non zero section (exercise)

we build a vector bundle ξ over W with fiber dimension 2 and $\xi|_M = \nu(M)$ such that $e(\xi)=0$ $\therefore e(\nu(M))=0$ and done by exercise above.

let $\eta = \pi^* \nu(M)$ be the pull-back of $\nu(M)$ to $\nu(M)$

$$\begin{tikzcd} \eta = \pi^*\nu(M) \arrow[r] \arrow[d] & \nu(M) \arrow[d, "\pi"] \\ \nu(M) \arrow[r, "\pi"] & M \end{tikzcd}$$

recall $\pi^*(M) = \{(a,b) \in \nu(M) \times \nu(M) : \pi(a) = \pi(b)\}$

define $s : \nu(M) \to \eta : e \mapsto (e,e)$

identify M with the zero section $Z \subset \nu(M)$

note $s \neq 0$ on $\nu(M) - Z$

so η is trivial on $\nu(M) - Z$

from differential topology we know that $M \subset W$ has a nbhd $N(M)$ diffeomorphic to a nbhd of Z in $\nu(M)$

so define ξ on $N(M)$ to be η and extend to rest of W via the trivial bundle

now s extends to ξ to be nonzero except on $M \subset N(M)$ $(\cong Z \subset \nu(M))$

so $e(\xi) = $ Poincaré Dual $([s^{-1}(\text{zero section})])$

$= [M] = 0$

since $\nu(M) = \xi|_M$ (exercise)

we have $e(\nu(M))$ has a nonzero section $\blacksquare$

For Th$^{\underline{m}}$ 3 proof need simple case of

Brown Representation Th$^{\underline{m}}$:

homotopy classes of maps

$$\boxed{\text{for a CW complex } H^n(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},n)]}$$

$K(\mathbb{Z},n)$ is a space such that

$$\pi_k(K(\mathbb{Z},n)) \cong \begin{cases} \mathbb{Z} & k=n \\ 0 & \text{otherwise} \end{cases}$$

eg. $K(\mathbb{Z},1) = S^1$ so

$$H^1(X) \cong [X, S^1]$$

Proof of Th$^{\underline{m}}$ 3:

let $N(M)$ be a tubular nbhd of M in W

by lemma 4, $N(M) \cong M \times D^2$

but there might be many such identifications

exercise: show the identifications are in one-to-one correspondence with $H^1(M)$

let $S^1 = \partial(\{p\} \times D^2)$ for any $p \in M$

no power of $S^1 \subset (W - N(M))$ is trivial in $H_2(W - N(M))$

since if it were there would be a 2-chain C s.t. $\partial C = nS^1$

so $C \cup n(\{p\} \times D^2)$ is a closed 2-chain and gives a homology class in $H_2(W)$ that intersects M, n times

but $[M] = 0$ so must intersect everything zero times

$\therefore [S^1]$ generates a $\mathbb{Z} \subset H_1(W - N(M))$

let $\alpha \in H^1(W-N(M))$ be its dual

now the inclusions

$$S^1 \overset{i}{\hookrightarrow} \partial(W-N(M)) \overset{j}{\hookrightarrow} W-N(M)$$

induce

$$\begin{array}{ccccc} H^1(W-N(M)) & \xrightarrow{j^*} & H^1(\partial(W-N(M))) & \xrightarrow{i^*} & H^1(S^1) \\ \cong & & \cong & & \cong \\ [W-N(M);S^1] & \to & [\partial(W-N(M));S^1] & \to & [S^1;S^1] \end{array}$$

restriction restriction

$j^*\alpha$ is represented by a map

$$f : \partial(W-N(M)) \to S^1$$

<u>exercise</u>: one can choose

$$h : \partial(W-N(M)) \to M \times S^1$$

such that

$$\begin{tikzcd} \partial(W-N(M)) \arrow[r, "f"] \arrow[d, "h"] & S^1 \\ M \times S^1 \arrow[ur, "\text{project}"'] & \end{tikzcd}$$

but now α is represented by $F : (W-N(M)) \to S^1$

s.t. $F|_\partial = f =$ projection

let $x \in S^1$ be a regular value

$\Sigma = F^{-1}(x)$ is a $(m+1)$-submanifold of $(W - N(M)) \subset W$

and $\partial\Sigma = M \times \{x\}$

$\uparrow$ isotopic to M so can isotop Σ s.t. $\partial\Sigma = M$

C. Homology of 3- and 4-manifolds

lemma 5:

If $X^4 = (0\text{-handle}) \cup k(2\text{-handles})$ then

$H_0(X) \cong \mathbb{Z}$, $H_1(X) = 0$ and

1) $H_2(X) \cong \oplus_k \mathbb{Z}$ generated by 2-handles

<u>specifically:</u> 2-handles attached to $L_1 \cup \ldots \cup L_k$, let Σ_i be a Seifert surface for L_k pushed into interior(B^4) and $A_i = \Sigma_i \cup C_i$ where C_i is the core of 2-handle

2) $H_2(X, \partial X) \cong \oplus_k \mathbb{Z}$ generated by the co-cores of the 2-handles

Proof:

1) let $X_i = (0\text{-handles}) \cup 1^{st}\ i$ (2-handles)

note $X_{i+1}/X_i \simeq S^2$

so
$$\underset{0}{\underset{\shortparallel}{H_3(X_{i+1}, X_i)}} \to H_2(X_i) \to H_2(X_{i+1}) \to \underset{A_{i+1} \longmapsto \pm 1 \in \mathbb{Z}}{\underset{\cong}{H_2(X_i, X_{i+1})}} \to \underset{0}{\underset{\shortparallel}{H_1(X_i)}}$$

$\therefore\ H_2(X_{i+1}) \cong H_2(X_i) \oplus \mathbb{Z}$ ← gen by A_{i+1}

so inductively $H_2(X)$ is as claimed.

2) since $H_1(X) = 0$, Universal Coefficients gives

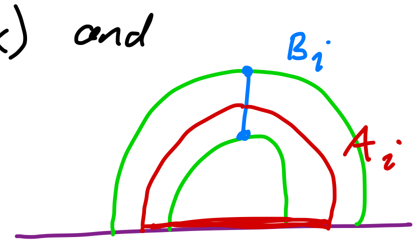
$$H^2(X) \cong H_2(X)$$

and Poincaré duality gives $H_2(X, \partial X) \cong H^2(X) \cong \oplus_k \mathbb{Z}$

note: if B_i is the cocore to i^{th} 2-handle

then $[B_i] \in H_2(X, \partial X)$ and

$$B_i \cdot A_j = \delta_{ij}$$



so B_i gives elt $H^2(X) \cong \mathrm{Hom}(H_2(X), \mathbb{Z})$

that is dual to A_i

i.e. B_i's generate $H_2(X, \partial X)$

Theorem 6:

let $L_1 \cup \ldots \cup L_k$ be a link in S^3 and X be the 4-mfd obtained from B^4 by attaching 2-handles to B^4 along the L_i with framing n_i

let $a_{ij} = \begin{cases} lk(L_i, L_j) & i \neq j \\ n_i & i = j \end{cases}$

the matrix $A = (a_{ij})$ is called the linking matrix

we have

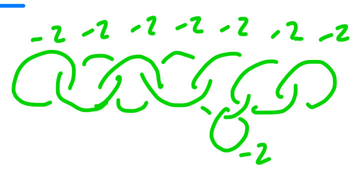
$$\begin{array}{ccccccc} H_2(X) & \longrightarrow & H_2(X,\partial X) & \xrightarrow{\partial} & H_1(\partial X) & \longrightarrow & H_1(X) \\ \wr\| & & \wr\| & & & & \| \\ \oplus_k \mathbb{Z} & \xrightarrow{A} & \oplus_k \mathbb{Z} & & & & 0 \end{array}$$

where we use A_i as a basis for $H_2(X)$ and
B_i " " $H_2(X,\partial X)$

Thus if μ_i is the meridian to L_i then we have a presentation for the homology of ∂X:

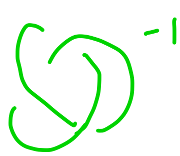
$$H_1(\partial X) \cong \langle \mu_1, \dots, \mu_n \mid a_{11}\mu_1 + a_{12}\mu_2 + \dots, \dots \rangle$$

example:

1) 

linking matrix $A = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & 0 & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 0 & 1 \\ & 0 & & & 1 & -2 & 1 & 0 \\ & & & & 0 & 1 & -2 & 0 \\ & & & & 1 & 0 & 0 & -2 \end{pmatrix}$ E_8 matrix

you can compute $\det A = 1$
so A is an isomorphism $\oplus_k \mathbb{Z} \to \oplus_k \mathbb{Z}$
and so $H_1(\partial X) = 1$

2) 

$A = [1]$ so $H_1(\partial X) = 1$

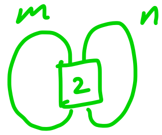
(of course ∂ in 1) & 2) are same as we showed earlier)

3) if $M^3 = n$-surgery on a knot then
$$H_1(M) = \mathbb{Z}_n$$

exercise:

1) if $X =$ 

, then see directly that any loop in ∂X is null-homologous in ∂X

2) let $X =$ 

what is $H_1(\partial M)$?

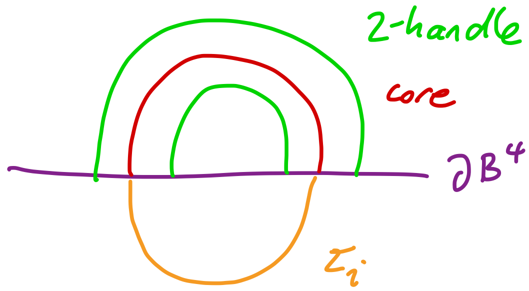
Proof: we first note that

$$H_2(X, \partial X) \longrightarrow H_1(\partial X)$$

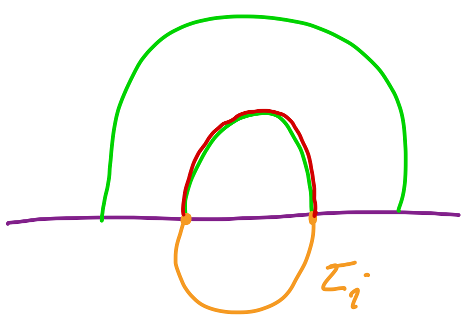
is just a boundary map so $\partial(B_i) = \mu_i$, so μ_i generate $H_1(\partial X)$

now $H_2(X) \to H_2(X, \partial X)$ is just inclusion so we need to write the A_i in terms of the B_i

recall $A_i = \Sigma_i \cup$ core

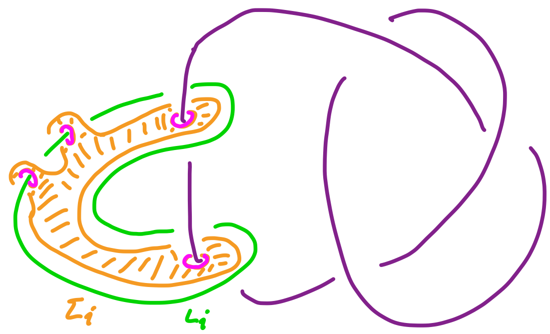


push core to ∂X



anything in ∂X "disapears" in $H_2(X, \partial X)$

so we just have Σ_i with its boundary linking L_i, n_i times in S^3



now push Σ_i into ∂X

can do this <u>except</u> for parts where some other component

$L_j \cap \Sigma_i$ and where $L_i \cap \Sigma_i$

these are precisely meridian curves bounding B_i in X

so $A_i \longmapsto a_{1i} B_1 + a_{2i} B_2 + \dots + a_{ki} B_k$

We now study $H_2(X)$ when X is closed

Recall $H_2(X) \cong H^2(X)$ (if $H_1(X)$ has no torsion)

and $H^2(X)$ has a cup-product pairing

$$H^2(X) \times H^2(X) \longrightarrow H^4(X) \cong \mathbb{Z}$$

we interperate this geometrically

lemma 7:

if X is a closed oriented 4-manifold then any $a \in H_2(X)$ is represented by an oriented surface $\Sigma \subset X$

ie. $[\Sigma] = a$

Proof: $H_2(X) \cong H^2(X) \underset{\text{Brown}}{\cong} [X; K(\mathbb{Z},2)]$

now $K(\mathbb{Z},2) = \mathbb{C}P^\infty = 0\text{-cell} \cup 2\text{-cell} \cup 4\text{-cell} \cup \dots$

so any map $f: X \to \mathbb{C}P^\infty$ is homotopic to

$$f: X \to \mathbb{C}P^2$$

(indeed, $f(X) \subset \mathbb{C}P^n$ some n since it is compact

now make f transverse to center of $2n$-cell

thus f disjoint from it if $n > 2$, and so f can be homotoped of of it, i.e. into $\mathbb{C}P^{n-1}$

so inductively get $f(X) \subset \mathbb{C}P^2$)

$H_2(\mathbb{C}P^2) \cong \mathbb{Z}$ generated by $\mathbb{C}P^1 \subset \mathbb{C}P^2$

make f transverse to $\mathbb{C}P^1$ and set $\Sigma = f^{-1}(\mathbb{C}P^1)$

$$\begin{tikzcd}
H_2(X) \arrow[d, "f_*"] \arrow[r, phantom, "\cong"] & H^2(X) \\
H_2(\mathbb{C}P^2) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & H^2(\mathbb{C}P^2) \arrow[u, "f^*"'] \arrow[d, phantom, "\cong"] \\
\mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

gen by $[\mathbb{C}P^1]$ $\qquad$ gen by P.D. $[\mathbb{C}P^1]$

so $P.D.(a) = f^*(P.D.[\mathbb{C}P^1]) = P.D.[f^{-1}(\mathbb{C}P^1)]$

i.e $a = [\Sigma]$

Big Question: given $a \in H_2(X)$ what is the minimal genus of a surface $\Sigma \subset X$ such that $[\Sigma] = a$?

given $[\Sigma], [\Sigma'] \in H_2(X^4)$

define $[\Sigma] \cdot [\Sigma'] =$ signed count of points in $\Sigma \cap \Sigma'$ (after they are made transverse) exercise: $[\Sigma]\cdot[\Sigma'] = \langle P.D.([\Sigma]) \cup P.D.([\Sigma']), [X]\rangle$

so the "intersection pairing"

$$H_2(X) \times H_2(X) \to \mathbb{Z}$$

and cup product pairing

$$H^2(X) \times H^2(X) \to \mathbb{Z}$$

are "dual"

in particular, by Poincaré Duality, it is non-degenerate

it is also symmetric and bilinear

denote it $Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$

lemma 8:

> If X is a 4-manifold made with only 0,2-handles (can also have 4-handle) and A is the linking matrix of the attaching circles of the 2-handles, then
>
> $$H_2(X) \times H_2(X) \to \mathbb{Z}$$
>
> is given by A in the basis A_i from lemma 5

Proof: recall $lk(L_i, L_j) = $ signed count $L_i \cap \tilde{\Sigma}_j$

Seifert surface in $S^3 = \partial B^4$

but this $=$ signed count $\Sigma_i \cap \tilde{\Sigma}_j$

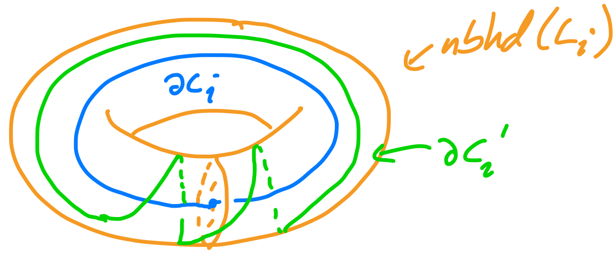
Seifert surface for L_i with interior pushed into B^4

$=$ signed count $\Sigma_i \cap \Sigma_j$

so $A_i \cap A_j = (C_i \cup \Sigma_i) \cap (C_j \cup \Sigma_j)$

$= \Sigma_i \cap \Sigma_j = lk(L_i, L_j)$

now for $A_i \wedge A_i$ take $(C_i \cup \Sigma_i)$ and $C_i' =$ parallel to C_i



since these link n_i times the surface Σ_i' that glues to C_i' intersects Σ_i, n_i times

$Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$ is an invariant of X

in general, invariants of non-degenerate symmetric bilinear forms $Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ are

1) rank$(Q) = r$

2) type — even if $Q(v,v)$ even $\forall v \in \mathbb{Z}^r$; odd otherwise

3) signature

$$\sigma(Q) = b_+ - b_-$$

$b_\pm$ = number of $\pm$ eigenvalues

4) definiteness

Q is positive definite if $b_- = 0$
negative definite if $b_+ = 0$
indefinite if $b_+ \neq 0 \neq b_-$

Algebraic Facts:

1) Q even then $\sigma(Q)$ divisible by 8

2) Q odd indefinite $\Rightarrow Q \cong \oplus^p (1) \oplus^q (-1)$ $p,q>0$

3) Q even indefinite $\Rightarrow Q \cong \pm \oplus^p E_8 \oplus^q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $q>0$

4) Q even definite $\Rightarrow$

$\sigma(Q)$	8	16	24	32	40	...
# of Q	1	2	24	7107	$>10^{51}$	...

lots!

5) Q odd definite $\Rightarrow$ even $\oplus \left(\pm \oplus^p (1)\right)$

Geometric Facts:

1) $X^4 = X_1 \cup_\partial X_2$ then $\sigma(X) = \sigma(X_1) + \sigma(X_2)$ $:= \sigma(Q_X)$

(note X_i not closed, can still define Q_{X_i} it is just not non-degenerate but still has signature)

2) $\sigma(X_1 \# X_2) = \sigma(X_1) + \sigma(X_2)$
(clear from 1))

3) $\sigma(-X) = -\sigma(X)$ (easy)
reverse orientation

4) X closed oriented 4-manifold then
$X = \partial W^5 \Leftrightarrow \sigma(X) = 0$

5) X closed, oriented, smooth 4-manifold with $\pi_1 = 1$ and Q_X even, then $\sigma(X)$ divisible by 16
(Rokhlin's Thm 1952)

6) If X is closed, oriented, smooth 4-manifold with $\pi_1 = 1$ and Q_X is definite, then $Q_X = \pm \oplus_k (1)$

(Donaldson 1983)

so the zoo of definite forms can be ignored when studying smooth 4-manifolds!

<u>Fact</u>: Every closed orientable 3-manifold bounds a 4-manifold $X = $ 0-handle $\cup$ 2-handles where the framings are even

(proof is just Kirby calculus, but a bit long)

now let M be a homology 3-sphere

(i.e. $H_*(M) \cong H_*(S^3)$)

let X be a 4-manifold as in fact above with $\partial X = M$

since M a homology sphere Q_X is non-degenerate

$\therefore$ Alg. fact 1) $\Rightarrow$ $\sigma(Q_X)$ is divisible by 8

set $\mu(M) = \sigma(X)/8 \mod 2$

<u>lemma 9</u>:

$\boxed{\mu(M) \text{ is well-defined}}$

<u>Proof</u>: need to see $\mu(M)$ is independent of X

so let X, X' be 2 such 4-manifolds

i.e. $Q_X, Q_{X'}$ even, simply connected, and $\partial X = M = \partial X'$

let $W = X \cup_M -X'$

W is a closed smooth 4-manifold with $\pi_1(W) = 1$

and Q_W is even

$\therefore$ Rokhlin (Geom. fact 5)) $\Rightarrow \sigma(Q_M)$ divisable by 16

that is $\sigma(Y) \underset{\uparrow}{=} \sigma(X) + \sigma(-X') \underset{\uparrow}{=} \sigma(X) - \sigma(X') \equiv 0 \mod 16$

geom fact 1)   geom fact 3)

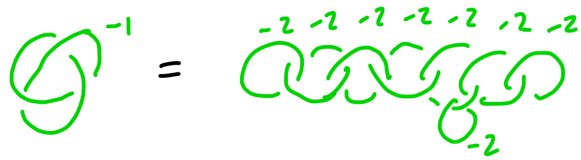
so $\sigma(X) \equiv \sigma(X') \mod 16$

and $\sigma(X)/8 \equiv \sigma(X')/8 \mod 2$

$\mu(M)$ is called the Rokhlin invariant of M

example:

1) P = Poincaré homology sphere

= 

$\leftarrow \sigma = -8$

so $\mu(P) = 1$

note: this implies P does not bound a homology 3-ball!

Determining when a homology 3-sphere bounds a homology ball is a major area of study

2) $\mu(S^3) = 0$

D. Kirby's Theorem

we are finally read to prove

Th^{m} 10 (Kirby 1978):

> manifolds obtained from two surgery diagrams in S^3 with integer surgery coefficients are
>
> diffeomorphic $\iff$ they are related by blowup/down and handle slides

Proof: ($\Leftarrow$) we have already shown this in lemma IV.4 and the discussion after it

($\Rightarrow$) let M_1 and M_2 be manifolds obtained from the surgery diagrams L_1 and L_2

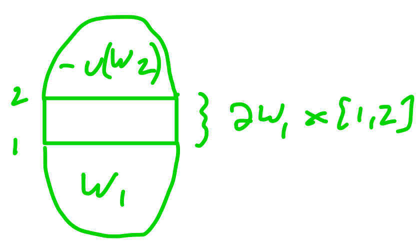
let $\phi : M_1 \to M_2$ be a diffeomorphism

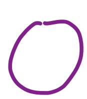
let W_i be the 4-manifold obtained from B^4 by attaching 2-handles to L_i

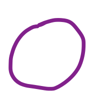
so $\partial W_i = M_i$

W_2 upside down

set $N = W_1 \cup_{id} ((\partial W_1) \times [1,2]) \cup_{\phi} (-\upsilon(W_2))$



exercise: 1)  $^{+1}$ $\cup$ 4-handle $\cong \mathbb{C}P^2$

 $^{-1}$ $\cup$ 4-handle $\cong \overline{\mathbb{C}P}^2$

2) taking connect sum of closed 4-manifolds corresponds to disjoint union of their handle pictures

3) $\sigma(\mathbb{CP}^2) = 1$, $\sigma(\overline{\mathbb{CP}}^2) = -1$

so by taking the connect sum of N with $\mathbb{CP}^2$'s or $\overline{\mathbb{CP}}^2$'s we can arrange $\sigma(N) = 0$

note: we can do this in W_1 part so we change the surgery picture for M_1 by blowups

by Geometric Fact 4) we know $\exists$ 5-manifold X such that $\partial X = N$

$\exists$ Morse function $f: M \to \mathbb{R}$ s.t.

- $f|_{\partial W_1 \times [0,1]} : \partial W_1 \times [0,1] \to [0,1]$ is projection
- $f^{-1}(1) = W_1$
- $f^{-1}(2) = W_2$
- no 0- and 5-handles

can think of X as $W_1 \times [1, 1+\varepsilon] \cup (1\text{-h})\text{'s} \cup (2\text{-h})\text{'s} \cup (3\text{-h})\text{'s} \cup (4\text{-h})\text{'s}$

just as in 4D we can exchange 1-handles for 2-handles without changing boundary

similarly we can exchange 4-handles for 3-handles without changing boundary

so can assume X only has 2- and 3-handles let the 2-handles be in $f^{-1}([1,1+2\varepsilon])$ and the 3-handles be in $f^{-1}([2-2\varepsilon,2])$

and set $W=f^{-1}(1.5)$

we get $f^{-1}([1,1.5])$ from $W_1\times[1,1+\varepsilon]=f^{-1}([1,1+\varepsilon])$

by attaching 2-handles:

$$h^2=D^2\times D^3 \quad \text{alond } \partial_-h^2=(\partial D^2)\times D^3$$

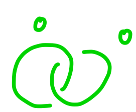
on the level of the boundary, we remove $S^1\times D^3$ and glue in $D^2\times S^2$

since the attaching sphere $S^1\times\{0\}$ in $W_1\times\{1+\varepsilon\}$ is null-homotopic $(\pi_1(W_1)=1)$ (and in dim 4, homotopy implies isotopy) can assume it bounds a disk in $B^4\subset W_1$

there are 2 framings $\pi_1(SO(3))=\mathbb{Z}/2$

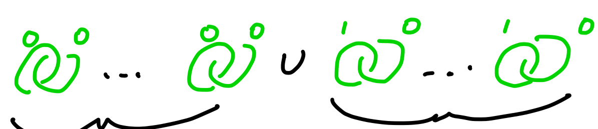
exercise:

1) this changes $W_1\times\{1+\varepsilon\}$ by connect sum with $S^2\times S^2$ or $S^2\tilde{\times}S^2$ ← twisted S^2 bundle over S^2

2)  $\cup$ 4-handle $=S^2\times S^2$

 $\cup$ 4-handle $=S^2\tilde{\times}S^2$

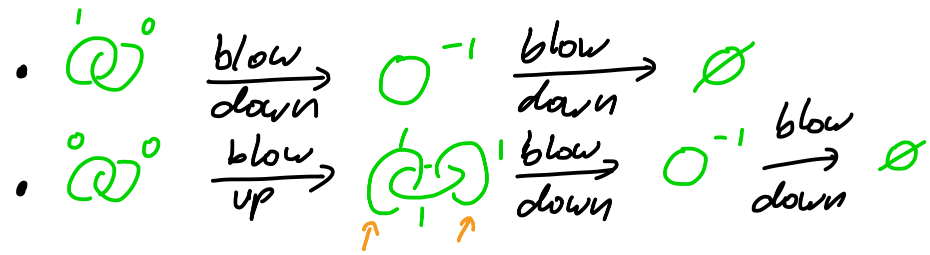
thus if L_1 is a handle diagram for W_1

then $L_1'=L_1\cup$ 

$S^2\times S^2$'s $\qquad S^2\tilde{\times}S^2$'s

is a link diagram for W

note: • $\partial W = M_1 = \partial W_1$



so surgery diagram L_1 for $M_1 = \partial W_1$ is related to the surgery diagram L_1' for ∂W by blowups and blowdowns

turn $f^{-1}([1.5, 2])$ upside down and we see
it is built from $W_2 \times [1, 1+\varepsilon]$ by attaching 2-handles so L_2 is related to a surgery diagram L_2' for ∂W by blowup and blowdowns

now we have two handle body diagrams for W
L_1' and L_2'

these correspond two Morse functions

$$f_i : W \to [-1, 1]$$

with only one index 0 crit pt in $(-1, 0)$
and index 2 critical points in $(0, 1)$

Cerf theory: $\exists$ 1-parameter family of functions

$$f_t : W \to [-1, 1] \quad t \in [1, 2]$$

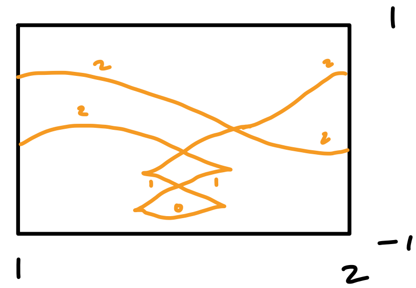
from f_1 to f_2 such that all f_t are Morse with distinct critical values except for a finite number of t's at these t's

① 2 critical points have same value

② birth-death of critical points

pairs: either 0,1 pair
1,2 pair
2,3 pair
3,4 pair

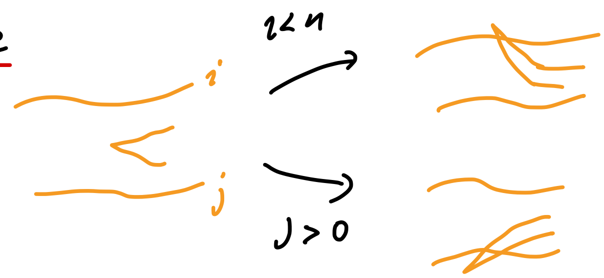
consider the Cerf graphic: plot the critical values of f_t in $[1,2]\times[-1,1]$



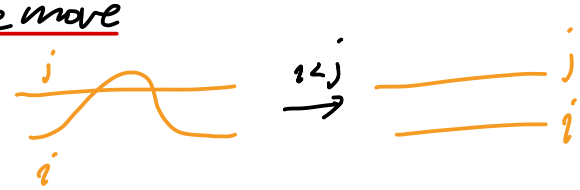
(#'s are index of critical points)

there are "moves" you can make to a Cerf diagram

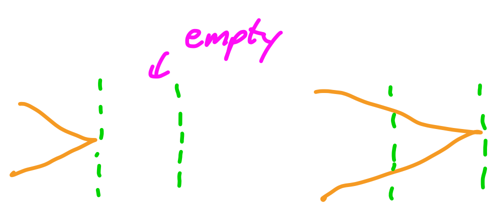
Beak move



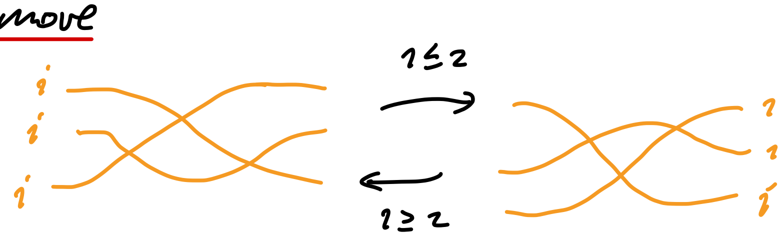
Independence move



beak isotopy



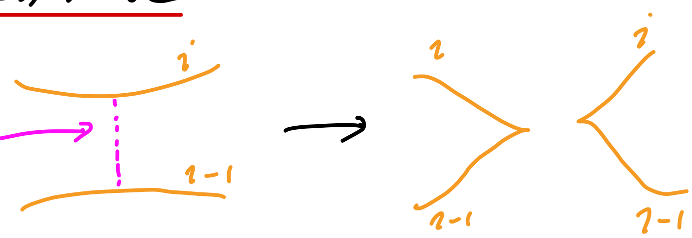
triangle move



Dove tail move

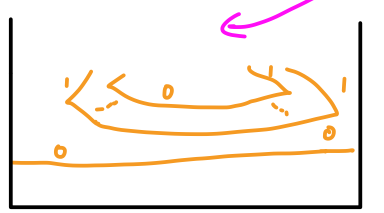
cancellation move

if these 2 handles can cancel



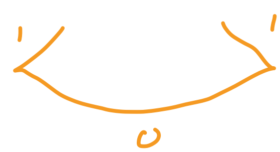
exercise: can assume graphic looks like

index 1 and above here



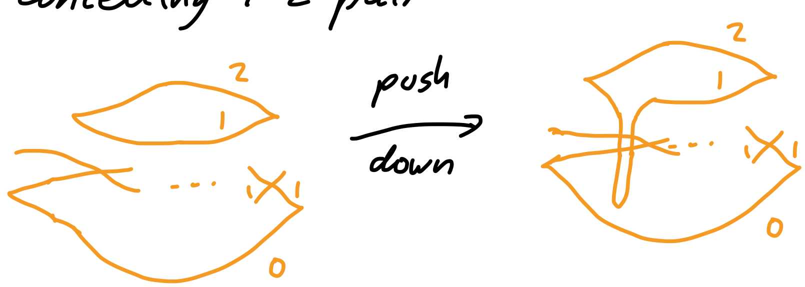
Claim: can elliminate extra index 0 critical points

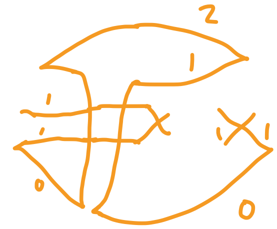
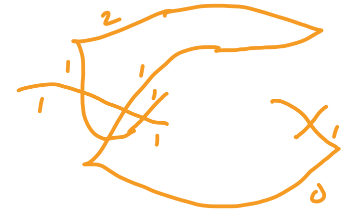
indeed: consider upper most

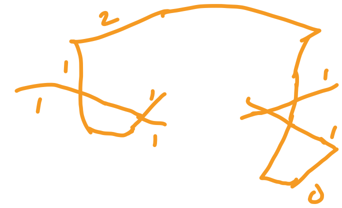


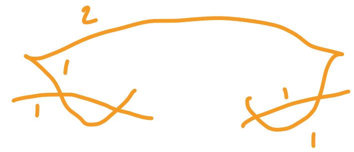
looks like

add cancelling 1-2 pair



$\xrightarrow[\text{move}]{\text{cancel}}$  $\xrightarrow[\text{tail}]{\text{dove}}$ 

$\xrightarrow[\text{of triangle moves}]{\text{lots}}$ 

$\xrightarrow[\text{tail}]{\text{dove}}$ 

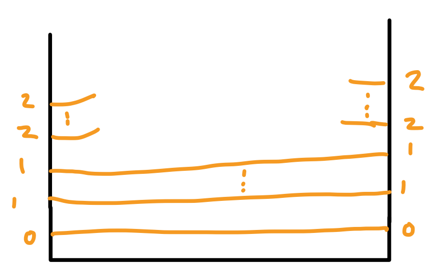
so we remove one 

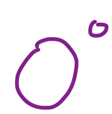
just keep doing this ✓

similarly, can remove all index 4-critical points

exercise: can arrange all index 1-critical points

now just push "beaks" of edge to get



recall we can replace  1-handle with  2-handle

just do this for all f_t at once to get

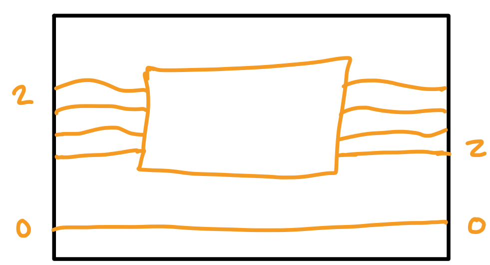
<u>exercise</u>: we know 

doesnt change ∂W, but show this change sends W to $W \# S^2 \times S^2$

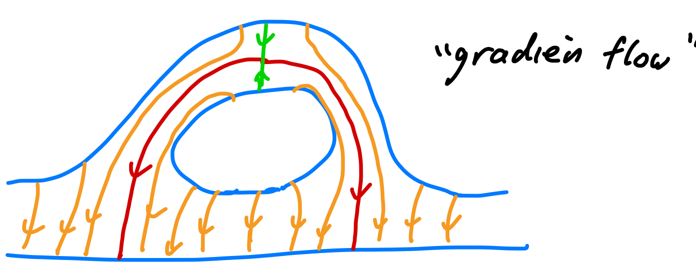
so the links L_1' and L_2' corresponding to f_1 and f_2 are changed by blowups and blow downs

we can similarly get rid of 3-handles

so we have



recall, a 2-handle seen through Morse f^n is

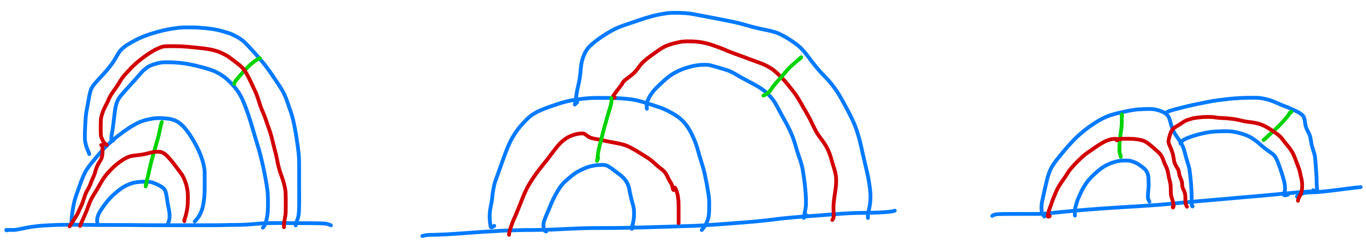


the red is the core extended by gradient flow

if the red (called the unstable manifold of the critical point) doesn't hit the green of another critical point then it reaches $\partial h^0 = S^3$

so if none of the red hit green then get a link in S^3, this is the handlebody diagram!

Smale: for all but finitely many t, the "red" and "green" are disjoint and at those finitely many t, the red hits the green in one point



such points give handle slides!

so as t increases the link L_t' associated to f_t is isotoped until a handle slide

so a finite # of handle slides from L_1' to L_2' and we are done $\boxplus$